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EXPRESSIONS FOR CERTAIN ACCELERATIONS OF A PARTICLE.

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The object of this paper is to present a method of obtaining expressions for the magnitudes of certain accelerations of a particle in terms of its speed and from our knowledge of its path. Let the path be plane and considered only at non-singular points, and let its equation be $f(x, y) = 0$. By differentiation with respect to time, we see that the motion of the particle satisfies the equation,

$$(1) \quad f'_x x' + f'_y y' = 0$$

where x', y' are time-derivatives. By differentiating (1) with respect to time and dividing by $\sqrt{f_x'^2 + f_y'^2}$, we have,

$$(2) \quad \frac{f''_x x'^2 + 2f''_{xy} x' y' + f''_y y'^2}{\sqrt{f_x'^2 + f_y'^2}} + \frac{f'_x x'' + f'_y y''}{\sqrt{f_x'^2 + f_y'^2}} = 0.$$

where x'', y'' are second time-derivatives. In the light of (1), we may put

$$\frac{f'_x x'' + f'_y y''}{\sqrt{f_x'^2 + f_y'^2}} = \frac{-y' x'' + x' y''}{\sqrt{x'^2 + y'^2}} = \alpha \cos \theta,$$

where α is the magnitude of the acceleration, and θ is the angle between the normal to the curve and the direction of acceleration. Hence for (2), we may write,

$$(3) \quad \frac{f''_x x'^2 + 2f''_{xy} x' y' + f''_y y'^2}{\sqrt{f_x'^2 + f_y'^2}} = -\alpha \cos \theta.$$

I. For example, let $f(x, y) = x^2 + y^2 - r^2$, and let the acceleration be directed toward the center of the circle. (3) now becomes the familiar formula:

$$(4) \quad \frac{x'^2 + y'^2}{\sqrt{x^2 + y^2}} = \alpha, \quad \text{or} \quad \alpha = \frac{v^2}{r}.$$

II. For another example, let $f(x, y) = b^2 x^2 + a^2 y^2 - a^2 b^2$, and let the acceleration be directed toward one focus F of the ellipse. For this example, (3) is

$$(5) \quad \frac{b^2 x'^2 + a^2 y'^2}{\sqrt{b^4 x^2 + a^4 y^2}} = -\alpha \cos \theta.$$

If r and r' be the focal distances from F and F' respectively to the particle at P , and p and p' be perpendiculars drawn from the foci to the tangent which has

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P for its point of tangency, elementary properties of the ellipse are that,

$$\cos \theta = -\frac{p}{r} = -\frac{p'}{r'} = -\sqrt{\frac{pp'}{rr'}} = -\frac{b}{\sqrt{rr'}} = \frac{-b}{\sqrt{a^2 - e^2 x^2}} = \frac{-ab^2}{\sqrt{b^4 x^2 + a^4 y^2}}.$$

And, since on the ellipse $b^2 x x' + a^2 y y' = 0$, the numerator in the first member of (5) has the value

$$\frac{b^4 x'^2}{y^2} = \frac{a^4 y'^2}{x^2}.$$

Hence (5) implies

$$(6_1) \quad x'^2 = \frac{a\alpha y^2}{b^2}, \quad (6_2) \quad y'^2 = \frac{b^2 \alpha x^2}{a^3};$$

whence

$$v^2 = \frac{\alpha(a^4 y^2 + b^4 x^2)}{a^3 b^2} = \frac{\alpha r r'}{a},$$

or

$$(7) \quad \alpha = \frac{a v^2}{r r'},$$

which is a generalization of (4).

By differentiating (6₁) as to time, and setting in the value of y' from (6₂), we have,

$$x'' = \frac{1}{2a} \frac{d\alpha}{dx} (a^2 - x^2) - \frac{x\alpha}{a}.$$

But the x -component of acceleration is

$$x'' = \alpha \frac{ae - x}{a - ex}.$$

From the comparison of the last two equations, we have

$$\frac{d\alpha}{2\alpha} = \frac{e dx}{a - ex}.$$

Whence the Newtonian law:

$$(8) \quad \alpha = \frac{k^2}{(a - ex)^2} = \frac{k^2}{r^2}.$$

A comparison of (7) and (8) yields the following elegant form of the *vis viva* equation:

$$v^2 = \frac{k^2}{a} \frac{r'}{r}.$$

The above discussion of elliptical motion, with only the slightest modification is valid for hyperbolic and parabolic motion under a central force directed toward a focus. And the last equation is valid in all three cases without modification, r'/a having the limiting value 2 in the case of the parabola.

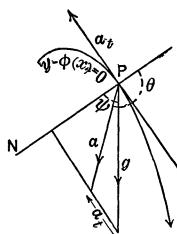
III. For the path of a projectile, we may take a parabola of higher order and let

$$f(x, y) = y - (a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n) = y - \varphi(x).$$

The range is assumed so short that the vertical direction may be regarded as constant. Now α is the resultant of the vertical gravitational acceleration $-g(y)$, and a tangential acceleration α_t . Obviously $\alpha \cos \theta = -g \cos \psi$, where ψ is the angle between the normal and the vertical. Hence (3) for this example is, by reference to (1),

$$-\frac{\varphi''(x)x^2}{\sqrt{f_x'^2 + f_y'^2}} = g \cos \psi = g \frac{dx}{\sqrt{dx^2 + dy^2}} = g \frac{f_y'}{\sqrt{f_x'^2 + f_y'^2}}.$$

or,



Therefore,

(9)

and,

$$x' = \frac{\sqrt{g}}{[-\varphi''(x)]^{1/2}}.$$

$$v = \frac{\sqrt{g}}{[-\varphi''(x)]^{1/2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

$$x'' = \frac{g}{2} \frac{\varphi'''(x)}{[\varphi''(x)]^2}.$$

But since the total horizontal acceleration is the horizontal component of α_t , it follows that

$$(10) \quad \alpha_t = \frac{g\varphi'''(x)}{2[\varphi''(x)]^2} \cdot \sqrt{1 + \left(\frac{dy}{dx}\right)^2};$$

or, by (9),

$$\alpha_t = v \frac{\sqrt{g} \varphi'''(x)}{2[-\varphi''(x)]^{3/2}}.$$

Now (9) and (10) can be written in the form $v = F_1(x)$ and $\alpha_t = F_2(x)$, where F_1 and F_2 are power series. The method of inversion of series gives

$$(11) \qquad \qquad \qquad F_2^{-1}(\alpha_t) = F_1^{-1}(v).$$

This entire discussion of the motion of a projectile is valid if the projectile is a finite sphere and also if the medium is not homogeneous. But the relation of α , to v depends on the character and condition of the medium. So, (11) has the nature of a law for the particular projectile if, and only if, the medium in which the flight occurred was equally resistant at all points in the observed path.